

# On the Automorphism Groups of Hyperbolic Manifolds<sup>\*†</sup>

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*We show that there does not exist a Kobayashi hyperbolic complex manifold of dimension  $n \neq 3$ , whose group of holomorphic automorphisms has dimension  $n^2 + 1$  and that, if a 3-dimensional connected hyperbolic complex manifold has automorphism group of dimension 10, then it is holomorphically equivalent to the Siegel space. These results complement earlier theorems of the authors on the possible dimensions of automorphism groups of domains in complex space.*

*The paper also contains a proof of our earlier result on characterizing  $n$ -dimensional hyperbolic complex manifolds with automorphism groups of dimensions  $\geq n^2 + 2$ .*

## 1 Introduction

Let  $M$  be a complex manifold of complex dimension  $n$  and  $\text{Aut}(M)$  the group of its holomorphic automorphisms. The group  $\text{Aut}(M)$  is a topological group with the natural compact-open topology. If  $M$  is Kobayashi-hyperbolic (e.g., if  $M$  is a bounded domain in complex space), then  $\text{Aut}(M)$  is in fact a finite-dimensional real Lie group whose topology agrees with the compact-open topology [Ko1], [Ko2].

We are interested in the problem of characterizing hyperbolic complex manifolds by the dimensions of their automorphism groups. Let  $M$  be such a manifold. It is known (see [Ka], [Ko1]) that  $\dim \text{Aut}(M) \leq n^2 + 2n$  and, if  $M$  is connected and  $\dim \text{Aut}(M) = n^2 + 2n$ , then  $M$  is holomorphically equivalent to the unit ball  $B^n \subset \mathbb{C}^n$ . In [IK] we obtained the following result.

**THEOREM 1.1** *Let  $M$  be a connected hyperbolic manifold of complex dimension  $n \geq 2$ . Then the following holds:*

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- (a) If  $\dim \operatorname{Aut}(M) \geq n^2 + 3$ , then  $M$  is biholomorphically equivalent to  $B^n$ .
- (b) If  $\dim \operatorname{Aut}(M) = n^2 + 2$ , then  $M$  is biholomorphically equivalent to  $B^{n-1} \times \Delta$ , where  $\Delta$  is the unit disc in  $\mathbb{C}$ .

In this paper we completely characterize hyperbolic manifolds with automorphism groups of dimension  $n^2 + 1$ . In [GIK] we observed that the automorphism group of a hyperbolic Reinhardt domain in  $\mathbb{C}^n$  cannot have dimension  $n^2 + 1$ . Therefore, it has been our expectation that there should be very few manifolds with  $\dim \operatorname{Aut}(M) = n^2 + 1$ . In fact, we have known of only one example of a manifold with such an automorphism group dimension; this occurs in dimension  $n = 3$ .

**Example.** Consider the 3-dimensional Siegel space (the symmetric bounded domain of type  $(III_2)$ ):

$$S := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : E - Z\bar{Z} \text{ is a positive-definite matrix}\},$$

where

$$Z := \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix},$$

and  $E$  is the  $2 \times 2$  identity matrix. The automorphism group of this domain is isomorphic to  $Sp_4(\mathbb{R})/\{\pm \operatorname{Id}\}$  and has dimension  $10 = 3^2 + 1$  (see, e.g., [S]).

In this paper we prove the following theorem that shows that the Siegel space  $S$  is indeed the only possibility.

**THEOREM 1.2** *Let  $M$  be a connected hyperbolic manifold of complex dimension  $n \geq 2$ . Then the following holds:*

- (a) If  $n \neq 3$ , then  $\dim \operatorname{Aut}(M) \neq n^2 + 1$ .
- (b) If  $n = 3$  and  $\dim \operatorname{Aut}(M) = 10$ , then  $M$  is holomorphically equivalent to the Siegel space  $S$ .

Note that, for  $n = 1$ , all manifolds with positive-dimensional automorphism groups are known explicitly [FK]. In particular, if  $\dim \operatorname{Aut}(M) = 3$ , then  $M$  is equivalent to  $\Delta$ , and there does not exist a hyperbolic manifold  $M$  with  $\dim \operatorname{Aut}(M) = 2$ .

There is probably no hope to obtain a complete classification of all  $n$ -dimensional hyperbolic manifolds with automorphism groups of dimension  $n^2$  or smaller. Indeed, in [GIK] we classified all hyperbolic Reinhardt domains in  $\mathbb{C}^n$  whose automorphism groups have dimension  $n^2$ , and even in this special case the classification is rather large and non-trivial.

For the sake of completeness of our exposition, and because some of the ideas originating there will recur in the proof of Theorem 1.2, we reproduce the proof of Theorem 1.1 in Section 2. We prove Theorem 1.2 in Section 3.

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## 2 Proof of Theorem 1.1

For the proof of Part (a) of Theorem 1.1 we need the following lemma.

**Lemma 2.1** *Let  $G$  be a Lie subgroup of the unitary group  $U(n)$  and let  $G^c$  be its connected component of the identity. Suppose that  $\dim G \geq n^2 - 2n + 3$ ,  $n \geq 2$ ,  $n \neq 4$ . Then either  $G = U(n)$ , or  $G^c = SU(n)$ . For  $n = 4$  this list must be augmented by subgroups of  $U(4)$  whose Lie algebras are isomorphic to  $\mathbb{R} \oplus \mathfrak{sp}_{2,0}$ .*

**Proof of Lemma 2.1:** Since  $G$  is compact, it is completely reducible (see, e.g., [VO]) and thus is isomorphic to a direct product  $G_1 \times \dots \times G_k$ ; here  $G_j$  for each  $j$  is a compact subgroup of  $U(n_j)$ ,  $\sum_{j=1}^k n_j = n$ , and  $G_j$  acts irreducibly on  $\mathbb{C}^{n_j}$ . Since  $\dim G_j \leq n_j^2$  and  $\dim G \geq n^2 - 2n + 3$ , it follows that  $k = 1$ , i.e.  $G$  acts (complex) irreducibly on  $\mathbb{C}^n$ .

Let  $\mathfrak{g} \subset \mathfrak{u}_n$  be the Lie algebra of  $G$  and  $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{gl}_n$  its complexification. It then follows that  $\mathfrak{g}^{\mathbb{C}}$  acts irreducibly and faithfully on  $\mathbb{C}^n$ . Therefore by a theorem of É. Cartan (see, e.g., [GG]),  $\mathfrak{g}^{\mathbb{C}}$  is either semisimple or is the direct sum of a semisimple ideal  $\mathfrak{h}$  and  $\mathbb{C}$ , where  $\mathbb{C}$  acts on  $\mathbb{C}^n$  by multiplication. Clearly the action of the ideal  $\mathfrak{h}$  on  $\mathbb{C}^n$  is irreducible and faithful.

Suppose first that  $\mathfrak{g}^{\mathbb{C}}$  is semisimple, and let  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_m$  be its decomposition into the direct sum of simple ideals. It then follows (see, e.g.,

[GG]) that the representation of  $\mathfrak{g}^{\mathbb{C}}$  is the tensor product of some irreducible faithful representations of  $\mathfrak{g}_j$ . Let  $n_j$  denote the dimension of the representation of  $\mathfrak{g}_j$ ,  $j = 1, \dots, m$ . Then  $n = n_1 \cdot \dots \cdot n_m$  and  $\dim_{\mathbb{C}} \mathfrak{g}_j \leq n_j^2 - 1$ ,  $n_j \geq 2$  for  $j = 1, \dots, m$ . It is now not difficult to prove the following claim.

**Claim:** If  $n = n_1 \cdot \dots \cdot n_m$ ,  $m \geq 2$ ,  $n_j \geq 2$  for  $j = 1, \dots, m$ , then  $\sum_{j=1}^m n_j^2 \leq n^2 - 2n$ .

It follows from the claim that  $m = 1$ , i.e.,  $\mathfrak{g}^{\mathbb{C}}$  is simple. The minimal dimensions of irreducible faithful representations of complex simple Lie algebras are well-known (see, e.g., [VO]). In the table below  $V$  denotes representations of minimal dimension.

$\mathfrak{g}$	$\dim V$	$\dim \mathfrak{g}$
$\mathfrak{sl}_k$ $k \geq 2$	$k$	$k^2 - 1$
$\mathfrak{o}_k$ $k \geq 7$	$k$	$\frac{k(k-1)}{2}$
$\mathfrak{sp}_{2k}$ $k \geq 2$	$2k$	$2k^2 + k$
$\mathfrak{e}_6$	27	78
$\mathfrak{e}_7$	56	133
$\mathfrak{e}_8$	248	248
$\mathfrak{f}_4$	26	52
$\mathfrak{g}_2$	7	14

Since  $\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}} \geq n^2 - 2n + 3$ , it follows that  $\mathfrak{g}^{\mathbb{C}} \simeq \mathfrak{sl}_n$ . Since  $\mathfrak{g}$  is a compact algebra, we get that  $\mathfrak{g} = \mathfrak{su}_n$  (see [VO]) and therefore  $G^c = SU(n)$  (note here that if  $\mathfrak{g}$  is a subalgebra in  $\mathfrak{u}_n$  and  $\mathfrak{g}$  is isomorphic to  $\mathfrak{su}_n$ , then  $\mathfrak{g}$  coincides with  $\mathfrak{su}_n$ , i.e., it consists exactly of matrices with zero trace).

Suppose now (for the second case) that  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} \oplus \mathbb{C}$ , where  $\mathfrak{h}$  is a semisimple ideal in  $\mathfrak{g}^{\mathbb{C}}$ . Then, repeating the above argument for  $\mathfrak{h}$  and taking into account that  $\dim \mathfrak{h} \geq n^2 - 2n + 2$ , we conclude that  $\mathfrak{h} \simeq \mathfrak{sl}_n$  for  $n \neq 4$ . Therefore, for  $n \neq 4$ ,  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{gl}_n$  and hence  $\mathfrak{g} = \mathfrak{u}_n$ , which implies that  $G = U(n)$ .

For  $n = 4$ , either  $\mathfrak{h} \simeq \mathfrak{sl}_4$  or  $\mathfrak{h} \simeq \mathfrak{sp}_4$ . We thus find that either  $\mathfrak{g} = \mathfrak{u}_4$  (in which case  $G = U(4)$ ), or  $\mathfrak{g} \simeq \mathbb{R} \oplus \mathfrak{sp}_{2,0}$ .

The lemma is proved.  $\square$

**Proof of Theorem 1.1, Part (a):** Let  $p \in M$  and let  $I_p$  denote the isotropy group of  $p$  in  $\text{Aut}(M)$ . Since the complex dimension of  $M$  is  $n$ , the

real dimension of any orbit of the action of  $\text{Aut}(M)$  on  $M$  does not exceed  $2n$ , and therefore we have  $\dim I_p \geq n^2 - 2n + 3$ .

Consider the isotropy representation  $\alpha_p : I_p \rightarrow GL(T_p(M), \mathbb{C})$ :

$$\alpha_p(f) := df(p), \quad f \in I_p.$$

The mapping  $\alpha_p$  is a continuous group homomorphism (see, e.g., Lemma 1.1 of [GK]) and thus is a Lie group homomorphism (see [Wa]). Since  $I_p$  is compact (see [Ko1]), there is a positive-definite Hermitian form  $h_p$  on  $T_p(M)$  such that  $\alpha_p(I_p) \subset U_{h_p}(n)$ , where  $U_{h_p}(n)$  is the group of complex linear transformations of  $T_p(M)$  preserving the form  $h_p$ . We choose a basis in  $T_p(M)$  such that  $h_p$  in this basis is given by the identity matrix.

By [E] and [Ki], the mapping  $\alpha_p$  is one-to-one. Further, since  $\dim I_p \geq n^2 - 2n + 3$ , we see that  $\alpha(I_p)$  is a compact subgroup of  $U_{h_p}(n)$  of dimension at least  $n^2 - 2n + 3$ . We are now going to use Lemma 2.1.

Assume first that  $n \neq 4$ . Then we have that either  $\alpha_p(I_p) = U_{h_p}(n)$ , or  $\alpha_p(I_p)^c = SU_{h_p}(n)$  (the latter denotes the subgroup of  $U_{h_p}(n)$  consisting of matrices with determinant 1). The groups  $U_{h_p}(n)$  and  $SU_{h_p}(n)$  act transitively on the unit sphere in  $T_p(M)$  and thus act transitively on directions in  $T_p(M)$  (see [GK] and [BDK] for terminology). Since  $M$  is non-compact (because the dimension of  $\text{Aut}(M)$  is positive—see [Ko1]), the main result of [GK] and its generalization in [BDK] applies. Thus  $M$  is biholomorphically equivalent to  $B^n$  (and therefore the possibility  $\alpha_p(I_p)^c = SU_{h_p}(n)$  is in fact not realizable).

Suppose now that  $n = 4$ . If we have that either  $\alpha_p(I_p) = U_{h_p}(4)$ , or else  $\alpha_p(I_p)^c = SU_{h_p}(4)$  for some  $p \in M$ , then by the above argument  $M$  is equivalent to  $B^4$ . Suppose now that the Lie algebra of  $\alpha_p(I_p)$  is isomorphic to  $\mathbb{R} \oplus \mathfrak{sp}_{2,0}$  for every  $p \in M$ . Then  $\dim \alpha_p(I_p) = 11$  for any  $p$ . Since  $\dim \text{Aut}(M) \geq 19$ , we have that in fact  $\dim \text{Aut}(M) = 19$ , and thus  $M$  is homogeneous. Therefore, by [N], [P-S],  $M$  is biholomorphically equivalent to a Siegel domain  $D \subset \mathbb{C}^4$  of the first or second kind. Further, we note that any representation  $\phi : \mathfrak{sp}_{2,0} \rightarrow \mathfrak{gl}_4$  is conjugate to the standard embedding of  $\mathfrak{sp}_{2,0}$  into  $\mathfrak{gl}_4$  by an element from  $GL(4, \mathbb{C})$  (to see this, one can extend  $\phi$  to a 4-dimensional representation of the complex Lie algebra  $\mathfrak{sp}_4$  and notice that such a representation is unique up to conjugation by elements of  $GL(4, \mathbb{C})$ ). Therefore  $\phi(\mathfrak{sp}_{2,0})$  contains an element  $X$  such that  $\exp(X) = -\text{id}$ , and thus  $\alpha_p(I_p)$  contains  $-\text{id}$  for any  $p$ . Hence the domain  $D$  is in fact symmetric. It

now follows from the explicit classification of symmetric Siegel domains (see [S]) that in fact there is no symmetric Siegel domain in  $\mathbb{C}^4$  with automorphism group of dimension equal to 19. This concludes the proof of Part **(a)**.

For the proof of Part **(b)** we will need the following lemma (which follows from the proof of Lemma 2.1).

**Lemma 2.2** *Let  $U_h(n)$  be the group of linear transformations of a complex  $n$ -dimensional space  $V$  that preserve a positive-definite Hermitian form  $h$  on  $V$ , and let  $G$  be a Lie subgroup of  $U_h(n)$  with  $\dim G \geq n^2 - 2n + 2$ ,  $n \geq 2$ ,  $n \neq 4$ . Then either  $G = U_h(n)$ , or  $G^c = SU_h(n)$ , or  $V$  splits into a sum of 1- and  $(n - 1)$ -dimensional  $h$ -orthogonal complex subspaces  $V^1$  and  $V^2$  such that  $G = U_{h^1}(1) \times U_{h^2}(n - 1)$ , where  $h^j$  is the restriction of  $h$  to  $V^j$ . For  $n = 4$ , there is the additional possibility that  $G$  can be any subgroup of  $U_h(4)$  with Lie algebra isomorphic to either  $\mathfrak{sp}_{2,0}$  or  $\mathbb{R} \oplus \mathfrak{sp}_{2,0}$ .*

**Proof of Theorem 1.1, Part (b):** We will use the notation from the proof of Part **(a)** above. Let  $p \in M$  and  $I_p$  be the isotropy group of  $p$  in  $\text{Aut}(M)$ . Then we have  $\dim I_p \geq n^2 - 2n + 2$  and thus  $\alpha_p(I_p)$  is a subgroup of  $U_{h_p}$  of dimension at least  $n^2 - 2n + 2$ . We now use Lemma 2.2. If, for some  $p \in M$ , we have that either  $\alpha_p(I_p) = U_{h_p}(n)$  or  $\alpha_p(I_p)^c = SU_{h_p}(n)$ , then  $\alpha_p(I_p)$  acts transitively on directions in  $T_p(M)$ . Hence, as in the proof of Part **(a)**,  $M$  is biholomorphically equivalent to  $B^n$ ; this is impossible since  $\dim \text{Aut}(M) = n^2 + 2$ .

Further suppose that, for any point  $p \in M$ ,  $T_p(M)$  splits into the sum of 1- and  $(n - 1)$ -dimensional  $h_p$ -orthogonal complex subspaces  $V_p^1$  and  $V_p^2$  such that  $\alpha_p(I_p) = U_{h_p^1}(1) \times U_{h_p^2}(n - 1)$ . In particular,  $\dim I_p = n^2 - 2n + 2$  for all  $p \in M$  and therefore  $M$  is homogeneous. Then, by [N], [P-S],  $M$  is biholomorphically equivalent to a homogeneous Siegel domain  $D$  of the first or second kind. Since  $\alpha_p(I_p)$  contains the transformation  $-\text{id}$  for all  $p \in M$ , the domain  $D$  is in fact symmetric. The theorem for  $n \neq 4$  now follows from the explicit classification of symmetric Siegel domains (see [S]).

Suppose now that  $n = 4$  and that, for some point  $p \in M$ , the Lie algebra of  $\alpha_p(I_p)$  is isomorphic to either  $\mathfrak{sp}_{2,0}$  or to  $\mathbb{R} \oplus \mathfrak{sp}_{2,0}$ . In the proof of Part **(a)** we noted that any embedding of  $\mathfrak{sp}_{2,0}$  into  $\mathfrak{gl}_4$  is conjugate by an element of  $GL(4, \mathbb{C})$  to the standard one. Therefore  $\alpha_p(I_p)$  contains a subgroup that is conjugate by an element of  $GL(4, \mathbb{C})$  to  $Sp_{2,0}$ . Since  $Sp_{2,0}$  acts transitively on the sphere of dimension 7, we get that  $\alpha_p(I_p)$  acts transitively on directions

in  $T_p(M)$  and therefore, as in the proof of Part **(a)**,  $M$  is biholomorphically equivalent to the unit ball which is impossible.

The theorem is proved.  $\square$

**Remark.** The argument in the last paragraph in the proof of Part **(b)** could also be used in the proof of Part **(a)** for the case  $n = 4$  without reference to the classification theory of symmetric domains. For hyperbolic Reinhardt domains, Theorem 1.1 was obtained by a different argument in [GIK].

### 3 Proof of Theorem 1.2

We will use the notation from the proof of Theorem 1.1. Suppose first that  $n = 2$ . Since  $\dim \operatorname{Aut}(M) = 5$ , for any point  $p \in M$  we have  $\dim \alpha_p(I_p) \geq 1$ . It follows from Lemma 2.2 that, if  $G$  is a positive-dimensional closed subgroup of  $U(2)$ , then one of the following holds:

(i)  $G = U(2)$ ;

(ii)  $G^c = SU(2)$ ;

(iii)  $\mathbb{C}^2$  splits into a sum of two 1-dimensional orthogonal subspaces  $V^1$  and  $V^2$ , such that  $G = U_{h^1}(1) \times U_{h^2}(1)$ , where  $h^j$  is the restriction of the standard Hermitian form on  $\mathbb{C}^2$  to  $V^j$ ;

(iv)  $G$  is 1-dimensional.

If, for some  $p \in M$ ,  $\alpha_p(I_p)$  is as in **(i)** or **(ii)**, then  $M$  is holomorphically equivalent to  $B^2 \subset \mathbb{C}^2$ , which is impossible since  $\dim \operatorname{Aut}(M) = 5$ .

If, for every  $p \in M$ ,  $\alpha_p(I_p)$  is as the group  $G$  in **(iii)** (i.e.,  $T_p(M)$  splits into a sum of subspaces  $V^1$  and  $V^2$  as in **(iii)**), then, for every  $p \in M$ ,  $\alpha_p(I_p)$  contains the element  $-\operatorname{id}$ . Since  $M$  is hyperbolic, it is a complex metric Banach manifold (when equipped with the Kobayashi metric). It now follows from the results of [V] (see Theorem 17.16 in [U]) that  $M$  is homogeneous, which is again impossible.

If, for every  $p \in M$ ,  $\alpha_p(I_p)$  is 1-dimensional, then  $M$  is homogeneous and hence is equivalent to a bounded homogeneous domain in  $\mathbb{C}^2$  [N]. Therefore,  $M$  is equivalent to either  $B^2$  or  $\Delta^2$ . But such an equivalence is impossible since  $\dim \operatorname{Aut}(B^2) = 8$  and  $\dim \operatorname{Aut}(\Delta^2) = 6$ .

We now assume that  $M = M_1 \cup M_2$ , with  $M_1, M_2 \neq \emptyset$ , and, for  $p \in M_1$ ,  $\alpha_p(I_p)$  is as the group  $G$  in **(iii)** and, for  $p \in M_2$ ,  $\alpha_p(I_p)$  is 1-dimensional. Fix  $p_0 \in M_2$  and let  $\Omega$  be the orbit of  $p_0$  under  $\operatorname{Aut}(M)^c$ . Then  $\Omega$  is a homogeneous subdomain of  $M$ . Therefore,  $\Omega$  is holomorphically equivalent to either  $B^2$  or  $\Delta^2$ . Consider the restriction map  $\phi : \operatorname{Aut}(M)^c \rightarrow \operatorname{Aut}(\Omega)$ :

$$\phi : f \mapsto f|_{\Omega}.$$

Clearly,  $\phi$  is continuous and hence a Lie group homomorphism. It is also one-to-one by the uniqueness theorem. Therefore  $\operatorname{Aut}(\Omega)$  contains a (not necessarily closed) subgroup  $H$  of dimension 5 that acts transitively on  $\Omega$ . Since  $M_1 \neq \emptyset$ ,  $H$  contains a subgroup isomorphic to  $U(1) \times U(1)$ . It now follows from the explicit formulas for the automorphism groups of  $B^2$  and  $\Delta^2$  that such a subgroup  $H$  in fact does not exist. This proves Part **(a)** for  $n = 2$ .

Suppose now that  $n = 3$ . If  $\dim \operatorname{Aut}(M) = 10$  then, for any  $p \in M$ , we have  $\dim \alpha_p(I_p) \geq 4$ . It follows from Lemma 2.2 that, if  $G$  is a closed subgroup of  $U(3)$  of dimension at least 4, then one of the following holds:

**(i)**  $G = U(3)$ ;

**(ii)**  $G^c = SU(3)$ ;

**(iii)**  $\mathbb{C}^3$  splits into a sum of a 1- and 2-dimensional orthogonal subspaces  $V^1$  and  $V^2$  respectively such that  $G = U_{h^1}(1) \times U_{h^2}(2)$ , where  $h^j$  is the restriction of the standard Hermitian form on  $\mathbb{C}^3$  to  $V^j$ ;

**(iv)**  $G$  is 4-dimensional.

If, for some  $p \in M$ ,  $\alpha_p(I_p)$  is as in **(i)** or **(ii)** then, as above,  $M$  is holomorphically equivalent to  $B^3 \subset \mathbb{C}^3$ , which is impossible since  $\dim \operatorname{Aut}(M) = 10$ .



If, for every  $p \in M$ ,  $\alpha_p(I_p)$  is as the group  $G$  in **(iii)**, then, by [V], as in the case  $n = 2$  above,  $M$  is homogeneous which is impossible.

If, for every  $p \in M$ ,  $\alpha_p(I_p)$  is 4-dimensional, then  $M$  is homogeneous and hence is holomorphically equivalent to one of the following domains:  $B^3$ ,  $B^2 \times \Delta$ ,  $\Delta^3$ , or the Siegel space  $S$ . Among these domains only  $S$  has automorphism group of dimension 10.

We now assume that  $M = M_1 \cup M_2$ , with  $M_1, M_2 \neq \emptyset$ , and, for  $p \in M_1$ , we suppose that  $\alpha_p(I_p)$  is as the group  $G$  in **(iii)** and, for  $p \in M_2$ ,  $\alpha_p(I_p)$  is 4-dimensional. As in the case  $n = 2$  above, this implies that there exists a subdomain  $\Omega \subset M$  such that  $\text{Aut}(\Omega)$  contains a subgroup  $H$  of dimension 10 that acts transitively on  $\Omega$  and that contains a subgroup isomorphic to  $U(1) \times U(2)$ . It now follows from the explicit formulas for the automorphism groups of  $B^3$ ,  $B^2 \times \Delta$  and  $S$  that such a subgroup  $H$  in fact does not exist. This proves Part **(b)**.

Let now  $n \geq 4$ . Since  $\dim \text{Aut}(M) = n^2 + 1$ , for any  $p \in M$  we have  $\dim \alpha_p(I_p) \geq n^2 - 2n + 1$ . It follows from Lemma 2.2 that, if  $G$  is a closed subgroup of  $U(n)$  of dimension at least  $n^2 - 2n + 1$ , then one of the following holds:

**(i)**  $G = U(n)$ ;

**(ii)**  $G^c = SU(n)$ ;

**(iii)**  $\mathbb{C}^n$  splits into a sum of a 1- and  $(n - 1)$ -dimensional orthogonal subspaces  $V^1$  and  $V^2$  respectively such that  $G = U_{h^1}(1) \times U_{h^2}(n - 1)$ , where  $h^j$  is the restriction of the standard Hermitian form on  $\mathbb{C}^n$  to  $V^j$ ;

**(iv)**  $\dim G = n^2 - 2n + 1$ ;

and, for  $n = 4$ , there is one more possibility:

**(v)** The Lie algebra of  $G$  is isomorphic to either  $\mathfrak{sp}_{2,0}$  or  $\mathbb{R} \oplus \mathfrak{sp}_{2,0}$ .

If, for some  $p \in M$ ,  $\alpha_p(I_p)$  is as in **(i)**, **(ii)** or **(v)**, then, as above,  $M$  is holomorphically equivalent to  $B^n \subset \mathbb{C}^n$ , which is impossible since

$\dim \operatorname{Aut}(M) = n^2 + 1$ .

If, for every  $p \in M$ ,  $\alpha_p(I_p)$  is as the group  $G$  in (iii), then by [V],  $M$  is homogeneous which is impossible.

If, for every  $p \in M$ ,  $\alpha_p(I_p)$  is  $n^2 - 2n + 1$ -dimensional, then  $M$  is homogeneous and hence, by [N], [P-S], is equivalent to a Siegel domain of the first or second kind in  $\mathbb{C}^n$ . We now need the following proposition which is of independent interest as it gives an alternative proof of Theorem 1.1 and a proof of Theorem 1.2 for Siegel domains in  $\mathbb{C}^n$ ,  $n \geq 4$ .

**Proposition 3.1** *Let  $U \subset \mathbb{C}^n$ ,  $n \geq 4$ , be a Siegel domain of the first or second kind. Suppose that  $\dim \operatorname{Aut}(U) \geq n^2 + 1$ . Then  $U$  is holomorphically equivalent to either  $B^n$  or  $B^{n-1} \times \Delta$ .*

**Proof of Proposition 3.1:** The domain  $U$  has the form

$$U = \left\{ (z, w) \in \mathbb{C}^{n-k} \times \mathbb{C}^k : \operatorname{Im} w - F(z, z) \in C \right\}, \quad (3.1)$$

where  $1 \leq k \leq n$ ,  $C$  is an open convex cone in  $\mathbb{R}^k$  not containing an entire line and  $F = (F_1, \dots, F_k)$  is a  $\mathbb{C}^k$ -valued Hermitian form on  $\mathbb{C}^{n-k} \times \mathbb{C}^{n-k}$  such that  $F(z, z) \in \overline{C} \setminus \{0\}$  for all non-zero  $z \in \mathbb{C}^{n-k}$ .

We will first show that  $k \leq 2$ . It follows from [KMO] that

$$\dim \operatorname{Aut}(U) \leq 4n - 2k + \dim \mathfrak{g}_0(U).$$

Here  $\mathfrak{g}_0(U)$  is the Lie algebra of all vector fields on  $\mathbb{C}^n$  of the form

$$X_{A,B} = Az \frac{\partial}{\partial z} + Bw \frac{\partial}{\partial w},$$

where  $A \in \mathfrak{gl}_{n-k}$ ,  $B$  belongs to the Lie algebra  $\mathfrak{g}(C)$  of the group  $G(C)$  of linear automorphisms of the cone  $C$ , and the following holds:

$$F(Az, z) + F(z, Az) = BF(z, z), \quad (3.2)$$

for  $z \in \mathbb{C}^{n-k}$ . By the definition of Siegel domain, there exists a positive-definite linear combination  $R$  of the components of the Hermitian form  $F$ . Then, for a fixed matrix  $B$  in formula (3.2), the matrix  $A$  is determined at most up to a matrix that is Hermitian with respect to  $R$ . Since the dimension

of the algebra of matrices Hermitian with respect to  $R$  is equal to  $(n - k)^2$ , we have

$$\dim \mathfrak{g}_0(U) \leq (n - k)^2 + \dim \mathfrak{g}(C),$$

and thus the following holds

$$\dim \text{Aut}(U) \leq 4n - 2k + (n - k)^2 + \dim \mathfrak{g}(C). \quad (3.3)$$

**Lemma 3.2** *We have*

$$\dim \mathfrak{g}(C) \leq \frac{k^2}{2} - \frac{k}{2} + 1.$$

**Proof of Lemma 3.2:** Fix a point  $x_0 \in C$  and consider its stabilizer  $G_{x_0}(C) \subset G(C)$ . The stabilizer is compact since it leaves stable the bounded open set  $C \cap (x_0 - C)$  and therefore we can assume that it is contained in the group  $O(k, \mathbb{R})$ . The group  $O(k, \mathbb{R})$  acts transitively on the sphere  $S(|x_0|)$  of radius  $|x_0|$  in  $\mathbb{R}^k$ , and the stabilizer  $H_{x_0}$  of the point  $x_0 \in S(|x_0|)$  under this action is isomorphic to  $O(k - 1, \mathbb{R})$ . Since  $G_{x_0} \subset H_{x_0}$ , we have

$$\dim G_{x_0} \leq \dim H_{x_0} = \frac{k^2}{2} - \frac{3k}{2} + 1,$$

and therefore

$$\dim \mathfrak{g}(C) \leq \frac{k^2}{2} - \frac{k}{2} + 1.$$

The lemma is proved. □

It now follows from (3.3) and Lemma 3.2 that

$$\dim \text{Aut}(U) \leq \frac{3k^2}{2} - k \left( 2n + \frac{5}{2} \right) + n^2 + 4n + 1. \quad (3.4)$$

It is easy to check that the right-hand side in (3.4) is strictly less than  $n^2 + 1$  for  $n \geq 4$  and  $k \geq 3$ . Therefore,  $k \leq 2$ .

If  $k = 1$ , the domain  $U$  is equivalent to  $B^n$ . Suppose that  $k = 2$ . Without loss of generality we can assume that the first component  $F_1$  of the  $\mathbb{C}^2$ -valued Hermitian form  $F$ , is positive-definite. We will show that the second component  $F_2$  has to be proportional to  $F_1$ . Indeed, if  $F_2$  is not proportional

to  $F_1$ , then in formula (3.2) the matrix  $A$  is determined by the matrix  $B$  up to transformations that are : (1) Hermitian with respect to the positive-definite form  $F_1$ ; (2) Hermitian with respect to some other Hermitian form which is not proportional to  $F_1$ . The dimension of the algebra of matrices satisfying conditions (1) and (2) does not exceed  $(n-2)^2 - 2$ , and therefore

$$\dim \mathfrak{g}_0(U) \leq (n-2)^2 - 2 + \dim \mathfrak{g}(C).$$

Hence

$$\dim \text{Aut}(U) \leq n^2 - 2 + \dim \mathfrak{g}(C),$$

which together with Lemma 3.2 implies

$$\dim \text{Aut}(U) \leq n^2,$$

which is a contradiction. Thus,  $F_2$  is proportional to  $F_1$ . Therefore,  $U$  is holomorphically equivalent to one of the following domains:

$$U_1 := \left\{ (z, w) \in \mathbb{C}^{n-2} \times \mathbb{C}^2 : \text{Im } w_1 - |z|^2 > 0, \text{Im } w_2 > 0 \right\},$$

or

$$U_2 := \left\{ (z, w) \in \mathbb{C}^{n-2} \times \mathbb{C}^2 : \text{Im } w_1 - |z|^2 > 0, \text{Im } w_2 - |z|^2 > 0 \right\}.$$

The domain  $U_1$  is equivalent to  $B^{n-1} \times \Delta$ . We will show that  $\dim \text{Aut}(U_2) < n^2 + 1$ . Let  $\mathfrak{g}(U_2)$  be the Lie algebra of  $\text{Aut}(U_2)$ . By [KMO],  $\mathfrak{g}(U_2)$  is a graded Lie algebra:

$$\mathfrak{g}(U_2) = \mathfrak{g}_{-1}(U_2) \oplus \mathfrak{g}_{-1/2}(U_2) \oplus \mathfrak{g}_0(U_2) \oplus \mathfrak{g}_{1/2}(U_2) \oplus \mathfrak{g}_1(U_2),$$

where  $\dim \mathfrak{g}_{-1}(U_2) = 2$ ,  $\dim \mathfrak{g}_{-1/2}(U_2) = 2(n-2)$ , and  $\mathfrak{g}_0(U_2)$  is described by (3.2). It is clear from (3.2) that  $\dim \mathfrak{g}_0(U_2) = n^2 - 4n + 5$ . Further,  $\mathfrak{g}_{1/2}(U_2)$  and  $\mathfrak{g}_1(U_2)$  admit explicit descriptions (see [S]). These descriptions imply that

$$\mathfrak{g}_{1/2}(U_2) = \{0\}, \quad \mathfrak{g}_1(U_2) = \{0\},$$

and therefore

$$\dim \mathfrak{g}(U_2) = n^2 - 2n + 3 < n^2 + 1.$$

The proposition is proved.  $\square$

We will now finish the proof of Theorem 1.2. It follows from Proposition 3.1 that, if, for every  $p \in M$ ,  $\alpha_p(I_p)$  is  $n^2 - 2n + 1$ -dimensional, then  $M$  has to be equivalent to either  $B^n$  or  $B^{n-1} \times \Delta$  which is impossible since the dimensions of the automorphism groups of these domains are bigger than  $n^2 + 1$ .

We now assume that  $M = M_1 \cup M_2$ , with  $M_1, M_2 \neq \emptyset$ , and, for  $p \in M_1$ , we suppose that  $\alpha_p(I_p)$  is as the group  $G$  in **(iii)** and, for  $p \in M_2$ ,  $\alpha_p(I_p)$  is  $n^2 - 2n + 1$ -dimensional. As in the cases  $n = 2, 3$  above, this implies that there exists a subdomain  $\Omega \subset M$  such that  $\text{Aut}(\Omega)$  contains a subgroup  $H$  of dimension  $n^2 + 1$  that acts transitively on  $\Omega$  and that contains a subgroup isomorphic to  $U(1) \times U(n-1)$ . It now follows from the explicit formulas for the automorphism groups of  $B^n$  and  $B^{n-1} \times \Delta$  that such a subgroup  $H$  in fact does not exist. This proves Part **(a)** for  $n \geq 4$ .

The theorem is proved.  $\square$

## References

- [BDK] Bland, J., Duchamp, T., and Kalka, M., A characterization of  $\mathbb{CP}^n$  by its automorphism group, *Complex Analysis* (University Park, Pa, 1986), 60–65, *Lecture Notes In Mathematics* 1268, Springer-Verlag, 1987.
- [E] Eisenman, D. A., Holomorphic mappings into tight manifolds, *Bull. Amer. Math. Soc.* 76(1970), 46–48.
- [FK] Farkas, H. and Kra, I., *Riemann Surfaces*, Second Edition, Springer-Verlag, 1992.
- [GIK] Gifford, J. A., Isaev, A. V. and Krantz, S. G., On the dimensions of the automorphism groups of hyperbolic Reinhardt domains, *Illinois J. Math.*, to appear.
- [GG] Goto, M. and Grosshans, F., *Semisimple Lie Algebras*, Marcel Dekker, 1978.

- [GK] Greene, R. E. and Krantz, S. G., Characterization of complex manifolds by the isotropy subgroups of their automorphism groups, *Indiana Univ. Math. J.* 34(1985), 865–879.
- [IK] Isaev, A. V. and Krantz, S. G., Characterization of Reinhardt domains by their automorphism groups, Proc. of the 3rd Korean Several Complex Variables Symposium, Seoul 16-19 December, 1998; to appear in *J. Korean Math. Soc.*
- [Ka] Kaup, W., Reele Transformationsgruppen und invariante Metriken auf komplexen Räumen, *Invent. Math.* 3(1967), 43–70.
- [KMO] Kaup, W., Matsushima, Y. and Ochiai, T., On the automorphisms and equivalences of generalized Siegel domains, *Amer. J. Math.* 92(1970), 475–497.
- [Ki] Kiernan, P., On the relations between taut, tight and hyperbolic manifolds, *Bull. Amer. Math. Soc.* 76(1970), 49–51.
- [Ko1] Kobayashi, S., *Hyperbolic Manifolds and Holomorphic Mappings*, Marcel Dekker, New York, 1970.
- [Ko2] Kobayashi, S., *Hyperbolic Complex Spaces*, Springer-Verlag, Berlin, 1998.
- [N] Nakajima, K., Homogeneous hyperbolic manifolds and homogeneous Siegel domains, *J. Math. Kyoto Univ.* 25(1985), 269–291.
- [P-S] Pyatetskii-Shapiro, I., *Automorphic Functions and the Geometry of Classical Domains* (translated from Russian), Gordon and Breach, 1969.
- [S] Satake, I., *Algebraic Structures of Symmetric Domains*, Kanô Memorial Lectures 4, Princeton University Press, 1980.
- [U] Upmeyer, H. *Symmetric Banach Manifolds and Jordan  $C^*$ -Algebras*, North-Holland, 1985.
- [V] Vigué, J.-P., Les automorphismes analytiques isométriques d’une variété complexe normée, *Bull. Soc. Math. France* 110(1982), 49–73.

- [VO] Vinberg, E. and Onishchik, A., *Lie Groups and Algebraic Groups*, Springer-Verlag, 1990.
- [Wa] Warner, F. W., *Foundations of Differential Manifolds and Lie Groups*, Scott, Foresman & Co., Glenview, London, 1971.

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